Discrete Mathematics

Topic 05 — Enumeration

Lecture 03 — Combinations and Permutations

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Outline

- Permutations — taking ordered sequences from a collection without repetition.
- Combinations — taking unordered sequences from a collection without repetition.

Outline

Permutations

Definition 1 (Permutations)

A permutation is a (possible) rearrangement of objects.

- For example, there are 6 permutations of the letters *a, b, c*:

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

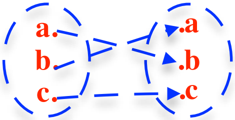
$\bar{3} * \bar{2} * \bar{1} = 3! = 6$

We know that we have them all listed above — there are 3 options for which letter we put first, then 2 options for which letter comes next, which leaves only 1 option for the last letter. The multiplication principle says we multiply $3 \times 2 \times 1$.

exactly one incomming arrow

exactly one outgoing arrow

- An equivalent definition is: A permutation is any bijective function on a finite set, i.e, source set and target set are the same and have finite number of elements, and the function is one-to-one and onto.



Example

Example 2

How many permutations are there of the letters *a, b, c, d, e, f*?

Solution. We do NOT want to try to list all of these out. However, if we did, we would need to pick a letter to write down first. There are 6 options for that letter. For each option of first letter, there are 5 options for the second letter (we cannot repeat the first letter; we are rearranging letters and only have one of each), and for each of those, there are 4 options for the third, 3 options for the fourth, 2 options for the fifth and finally only 1 option for the last letter.

So there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ permutations of the 6 letters.

Permutations of n elements

There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

permutations of n (distinct) elements.

Counting Permutations

In general, we can ask how many permutations exist of k objects choosing those objects from a larger collection of n objects where $k \leq n$.

Permutations of k-elements from a collection of n elements

The number of permutations of k elements taken from a set of n (distinct) elements is

$$\mathrm{P}(n,k) = (n) \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

- The number of different collections of k objects where **order matters** from a collection of n objects is $\mathrm{P}(n,k)$.
- Alternative notation: $\mathrm{P}(n,k) = {}^{n}\mathrm{P}_{k}$.
- $\mathrm{P}(n,k)$ is sometimes called the number of "k-permutations of n elements".
- $\mathrm{P}(n,n) = n!$, i.e., $k = n$

Example

Example 3 (Counting Bijective Functions)

How many functions $f : \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ are *bijective*[a]?

Solution. Each of the 8 elements in the source is mapped to a single distinct element in the target so the number of bijective functions is

$$8 \times 7 \times \cdots \times 1 = 8! = \mathrm{P}(8, 8)$$

[a] Each element in the source is mapped to each element in the target and vice-versa.

Example 4 (Counting injective functions)

How many functions $f : \{1, 2, 3\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ are *injective*?

Solution. Note that f cannot be a bijection here. Why?
Using the multiplication principle and using each element in target at most once, the number of injective functions is

$$8 \times 7 \times 6 = \frac{8!}{5!} = \frac{8!}{(8-3)!} = \mathrm{P}(8, 3)$$

Outline

Counting Combinations

If, the order does not matter when drawing k object from a larger collection of n distinct objects we are working with combinations rather than permutations.

Combinations of k-elements from a collection of n elements

The number of combinations of k elements taken from a set of n (distinct) elements is

$$C(n,k) = \frac{(n)\cdot(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k} = \frac{P(n,k)}{k!}$$

- The number of different collections of k objects where **order does not matters** from a collection of n objects is $C(n,k)$.
- Note the $k!$ in the denominator is to take account of duplicates due to ignoring the order.
- Alternative notation: $C(n,k) = {}^{n}C_{k} = \binom{n}{k}$.
- $C(n,k)$ is sometimes called the number of "k-combinations of n elements".
- $C(n,n) = C(n,0) = 1$, i.e., only one way to pick all elements, and only one way to pick zero elements.

Example 5 I

Example 5

I decide to have a dinner party. Even though, for a mathematician, I'm incredibly popular and have 14 different friends, I only have enough chairs to invite 6 of them.

(a) How many options do I have for which 6 friends to invite?

(b) What if I needed to decide not only which friends to invite but also where to seat them along my long table? How many options do I have then?

Solution.

(a) *How many options do I have for which 6 friends to invite?*
Here I need to pick 6 from a collection of 14 distinct objects. Order does not matter $\implies$ combinations.
This can be done in $\binom{14}{6} = 3003$ ways.

(b) *How many options ... to decide ... which friends to invite ... where to seat them ... ?*
Again, I need to pick 6 from a collection of 14 distinct objects. But here order does matter $\implies$ permutations.
So the answer is $P(14, 6) = 2.192.190$.

Review Exercises 1 (Combinations)

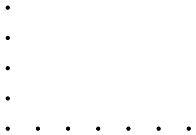
Question 1:
A pizza parlour offers 10 toppings.

- (a) How many 3-topping pizzas could they put on their menu? Assume double toppings are not allowed.
- (b) How many total pizzas are possible, with between zero and ten toppings (but not double toppings) allowed?
- (c) The pizza parluor will list the 10 toppings in two equal-sized columns on their menu. How many ways can they arrange the toppings in the left column?

Question 2:
Using the digits 2 through 8, find the number of different 5-digit numbers such that:

- (a) Digits can be used more than once.
- (b) Digits cannot be repeated, but can come in any order.
- (c) Digits cannot be repeated and must be written in increasing order.
- (d) Which of the above counting questions is a combination and which is a permutation? Explain why this makes sense.

Question 3:
How many triangles are there with vertices from the points shown below? Note, we are not allowing degenerate triangles — ones with all three vertices on the same line, but we do allow non-right triangles. Explain why your answer is correct.



Hint. You need exactly two points on either the x- or y-axis, but don't over-count the right triangles.