

Computational Thinking

# Discrete Mathematics

Number Theory

## Topic 05 — Enumeration

Logic

### Lecture 03 — Combinations and Permutations

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Graphs and  
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Collections

#### Outline

- Permutations — taking ordered sequences from a collection without repetition.
- Combinations — taking unordered sequences from a collection without repetition.

Enumeration

Relations & Functions

# Outline

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| 1. Permutations | 2 |
| 2. Combinations | 7 |

# Permutations

## Definition 1 (Permutations)

A **permutation** is a (possible) rearrangement of objects.

- For example, there are 6 permutations of the letters  $a, b, c$ :

$abc, acb, bac, bca, cab, cba.$

We know that we have them all listed above — there are 3 options for which letter we put first, then 2 options for which letter comes next, which leaves only 1 option for the last letter. The multiplication principle says we multiply  $3 \times 2 \times 1$ .

- An equivalent definition is: A **permutation** is any bijective function on a finite set, i.e, source set and target set are the same and have finite number of elements, and the function is one-to-one and onto.

# Example

## Example 2

How many permutations are there of the letters  $a, b, c, d, e, f$ ?

**Solution.** We do NOT want to try to list all of these out. However, if we did, we would need to pick a letter to write down first. There are 6 options for that letter. For each option of first letter, there are 5 options for the second letter (we cannot repeat the first letter; we are rearranging letters and only have one of each), and for each of those, there are 4 options for the third, 3 options for the fourth, 2 options for the fifth and finally only 1 option for the last letter.

So there are  $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$  permutations of the 6 letters.

Permutations of  $n$  elements

There are

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$$

permutations of  $n$  (distinct) elements.

# Counting Permutations

In general, we can ask how many permutations exist of  $k$  objects choosing those objects from a larger collection of  $n$  objects where  $k \leq n$ .

Permutations of  $k$ -elements from a collection of  $n$  elements

The number of permutations of  $k$  elements taken from a set of  $n$  (distinct) elements is

$$P(n, k) = (n) \cdot (n - 1) \cdots (n - k + 1) = \frac{n!}{(n - k)!}$$

- The number of different collections of  $k$  objects where **order matters** from a collection of  $n$  objects is  $P(n, k)$ .
- Alternative notation:  $P(n, k) = {}^n P_k$ .
- $P(n, k)$  is sometimes called the number of “ $k$ -permutations of  $n$  elements”.
- $P(n, n) = n!$ , i.e.,  $k = n$

# Example

## Example 3 (Counting Bijective Functions)

How many functions  $f : \{1, 2, \dots, 8\} \rightarrow \{1, 2, \dots, 8\}$  are *bijective*<sup>a</sup>?

**Solution.** Each of the 8 elements in the source is mapped to a single distinct element in the target so the number of bijective functions is

$$8 \times 7 \times \dots \times 1 = 8! = P(8, 8)$$

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<sup>a</sup>Each element in the source is mapped to each element in the target and vice-versa.

## Example 4 (Counting injective functions)

How many functions  $f : \{1, 2, 3\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8\}$  are *injective*?

**Solution.** Note that  $f$  cannot be a bijection here. Why?

Using the multiplication principle and using each element in target at most once, the number of injective functions is

$$8 \times 7 \times 6 = \frac{8!}{5!} = \frac{8!}{(8-3)!} = P(8, 3)$$

# Outline

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1. Permutations

2

2. Combinations

7

# Counting Combinations

If, the order does not matter when drawing  $k$  object from a larger collection of  $n$  distinct objects we are working with **combinations** rather than **permutations**.

Combinations of  $k$ -elements from a collection of  $n$  elements

The number of combinations of  $k$  elements taken from a set of  $n$  (distinct) elements is

$$C(n, k) = \frac{(n) \cdot (n - 1) \cdots (n - k + 1)}{k!} = \frac{n!}{(n - k)!k!} = \binom{n}{k} = \frac{P(n, k)}{k!}$$

- The number of different collections of  $k$  objects where **order does not matters** from a collection of  $n$  objects is  $C(n, k)$ .
- Note the  $k!$  in the denominator is to take account of duplicates due to ignoring the order.
- Alternative notation:  $C(n, k) = {}^nC_k = \binom{n}{k}$ .
- $C(n, k)$  is sometimes called the number of “ $k$ -combinations of  $n$  elements”.
- $C(n, n) = C(n, 0) = 1$ , i.e., only one way to pick all elements, and only one way to pick zero elements.



# Example 5

## Example 5

I decide to have a dinner party. Even though, for a mathematician, I'm incredibly popular and have 14 different friends, I only have enough chairs to invite 6 of them.

- (a) How many options do I have for which 6 friends to invite?
- (b) What if I needed to decide not only which friends to invite but also where to seat them along my long table? How many options do I have then?

### Solution.

- (a) *How many options do I have for which 6 friends to invite?*

Here I need to pick 6 from a collection of 14 distinct objects. Order does not matter  $\implies$  combinations.

This can be done in  $\binom{14}{6} = 3003$  ways.

- (b) *How many options ... to decide ... which friends to invite ... where to seat them ... ?*

Again, I need to pick 6 from a collection of 14 distinct objects. But here order does matter  $\implies$  permutations.

So the answer is  $P(14, 6) = 2.192.190$ .

# Review Exercises 1 (Combinations)

## Question 1:

A pizza parlour offers 10 toppings.

- (a) How many 3-topping pizzas could they put on their menu? Assume double toppings are not allowed.
- (b) How many total pizzas are possible, with between zero and ten toppings (but not double toppings) allowed?
- (c) The pizza parlour will list the 10 toppings in two equal-sized columns on their menu. How many ways can they arrange the toppings in the left column?

## Question 2:

Using the digits 2 through 8, find the number of different 5-digit numbers such that:

- (a) Digits can be used more than once.
- (b) Digits cannot be repeated, but can come in any order.
- (c) Digits cannot be repeated and must be written in increasing order.
- (d) Which of the above counting questions is a combination and which is a permutation? Explain why this makes sense.

## Question 3:

How many triangles are there with vertices from the points shown below? Note, we are not allowing degenerate triangles — ones with all three vertices on the same line, but we do allow non-right triangles. Explain why your answer is correct.



**Hint.** You need exactly two points on either the  $x$ - or  $y$ -axis, but don't over-count the right triangles.